Calculus I - Lecture 18 - Curve Sketching

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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March 31, 2014

Goal: Use first and second derivatives to make a rough sketch of the graph of a function $f(x)$.

Recall:

Critical points: Points c in the domain of $f(x)$ where $f'(c)$ does not exist or $f'(c) = 0$.

Monotony:

$f'(x) > 0 \Rightarrow f(x)$ increasing.
$f'(x) < 0 \Rightarrow f(x)$ decreasing.

Local extrema: Appear at points c in the domain of $f(x)$ where $f(x)$ changes from increasing to decreasing ($f(c)$ maximum) or from decreasing to increasing ($f(c)$ minimum).

First derivative test:

Let $f'(c) = 0$. Then:

$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c \Rightarrow f(c)$ is local max.
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c \Rightarrow f(c)$ is local min.

Concavity:

$f''(x) > 0 \Rightarrow f(x)$ concave up.
$f''(x) < 0 \Rightarrow f(x)$ concave down.

Inflection points: Points $(x, f(x))$ where the graph of $f(x)$ changes its concavity.

Inflection point test:

Let $f''(c) = 0$.
If $f''(x)$ changes its sign at $x = c$ then $f(x)$ has a inflection point at $x = c$.

Second derivative test:

Let $f'(c) = 0$. Then:

$f''(c) < 0 \Rightarrow f(c)$ is a local maximum
$f''(c) > 0 \Rightarrow f(c)$ is a local minimum

Transition points: Points where $f'(x)$ or $f''(x)$ has a sign change.

Those are the points where the graph of $f(x)$ may changes its features. We will concentrate to find those points

Graph Sketching

Main Steps

1. Determine then domain.
2. Find points with $f'(x) = 0$ and mark sign of $f'(x)$ on number line.
3. Find points with $f''(x) = 0$ and mark sign of $f''(x)$ on number line.
4. Compute function values for transition points.
5. Find asymptotes.
6. Sketch graph.
7. Take a break.

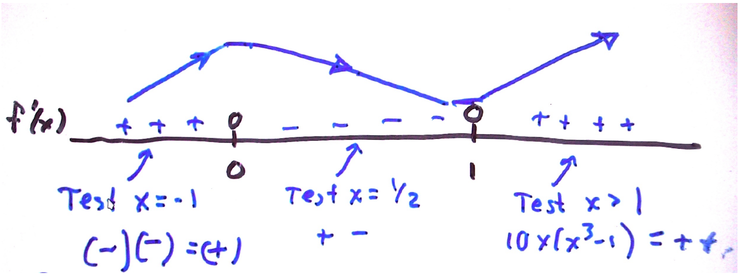
Example: Sketch the graph of $f(x) = 2x^5 - 5x^2 + 1$.

Solution:

1. All real numbers (polynomial).

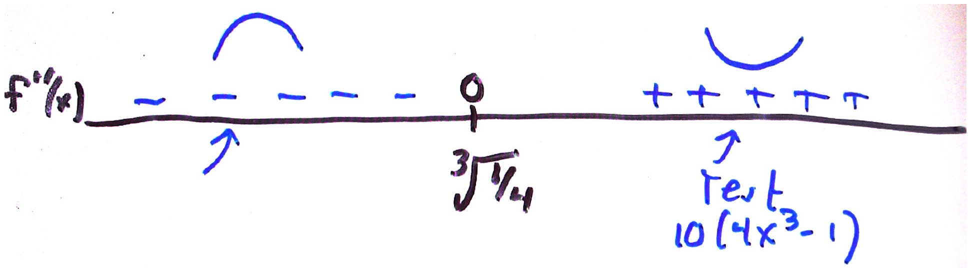
2. $f'(x) = 5 \cdot 2x^4 - 2 \cdot 5x = 10x(x^3 - 1)$

$f'(x) = 0$ when $x = 0$ or $x^3 = 1 \Leftrightarrow x = \sqrt[3]{1} = 1$.



3. $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}(10x^4 - 10x) = 4 \cdot 10x^3 - 10 = 10(4x^3 - 1)$

$f''(x)$ when $4x^3 = 1 \Leftrightarrow x^3 = 1/4 \Leftrightarrow x = 1/\sqrt[3]{4}$.

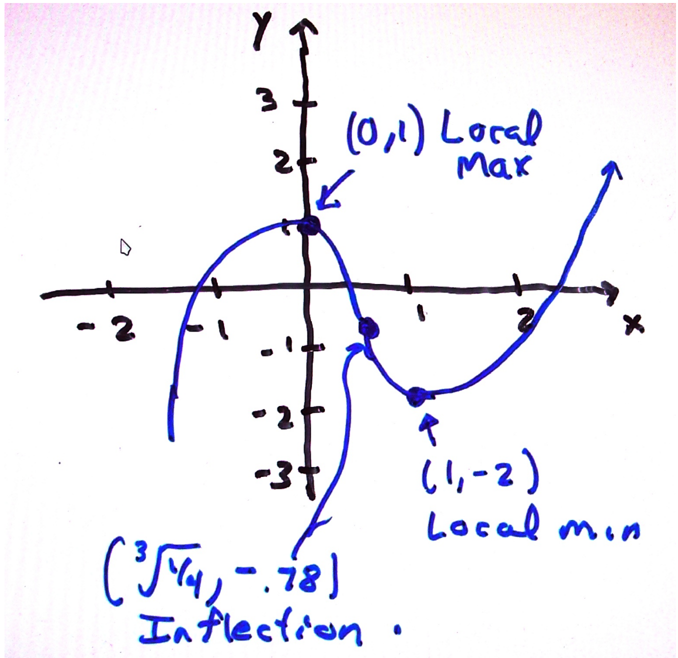


4.

x	$f(x) = 2x^5 - 5x^2 + 1$
0	1
1	$2 - 5 + 1 = -2$
$1/\sqrt[3]{4}$	$-.78$

5. No asymptotes (polynomial of positive degree)

6.



7. Questions?

Example: Sketch the graph of $f(x) = x(x-2)^{2/3}$.

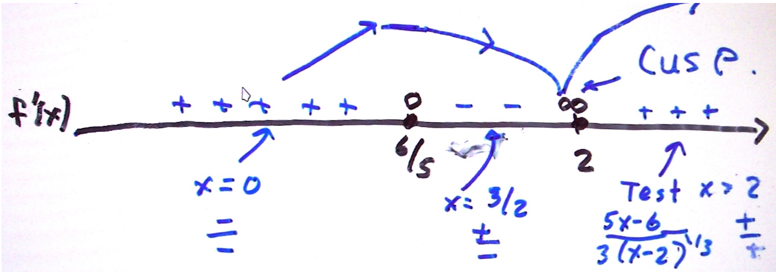
Solution:

1. All real numbers ($\sqrt[3]{x}$ is defined everywhere).
2. $f'(x) = (x-2)^{3/2} + x \cdot \frac{2}{3}(x-2)^{-1/3}$

$$= \frac{3(x-2)}{3(x-2)^{1/3}} + \frac{2x}{3(x-2)^{1/3}} = \frac{5x-6}{3(x-2)^{1/3}}$$

$f'(x) = 0 \Leftrightarrow 5x = 6 \Leftrightarrow x = 6/5$

$f'(x)$ D.N.E. at $x = 2$.

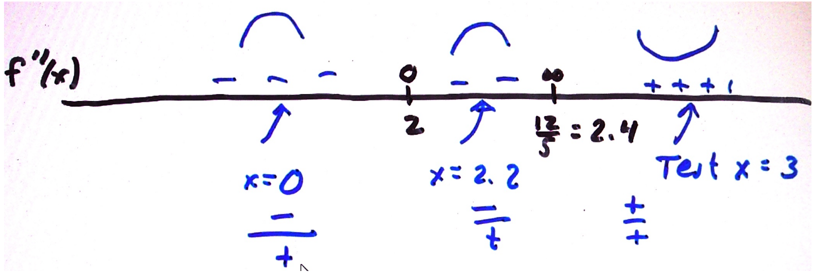


3. $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{5x-6}{3(x-2)^{1/3}}\right)$

$$= \frac{5 \cdot 3(x-2)^{1/3} - (5x-6)(x-2)^{-2/3}}{9(x-2)^{2/3}}$$

$$= \frac{15(x-2) - (5x-6)}{9(x-2)^{4/3}} = \frac{10x-24}{9(x-2)^{4/3}}$$

$f''(x) = 0 \Leftrightarrow 10x = 24 \Leftrightarrow x = 24/10 = 12/5$

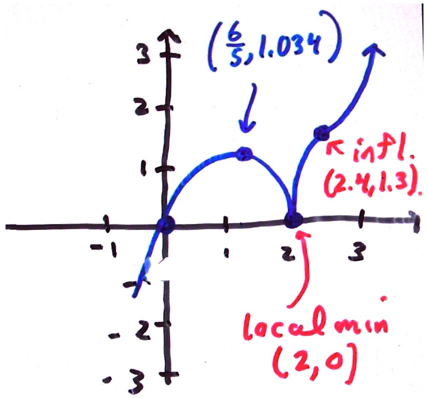


4.

x	$f(x) = x(x-2)^{2/3}$
$6/5$	1.034
2	0
2.4	1.3
0	0

5. No asymptotes

6.



Example: Sketch the graph of $f(x) = \dfrac{x}{x^2 - 9}$.

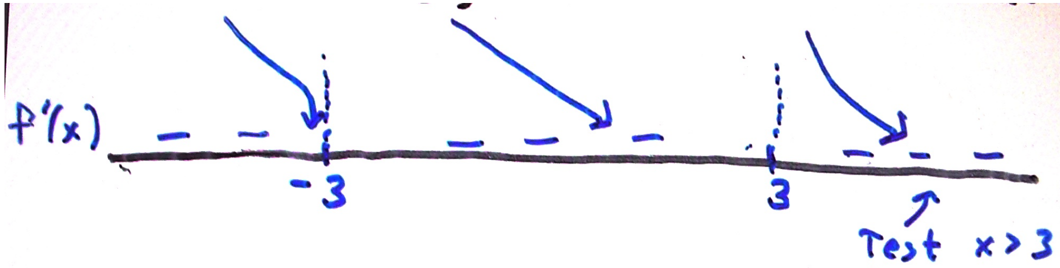
Solution:

1. Denominator $x^2 - 9 \neq 0$ which is $x \neq \pm 3$.

2. $f'(x) = \dfrac{(x^2 - 9) - x \cdot 2x}{(x^2 - 9)^2}$

$$\frac{-x^2 - 9}{(x^2 - 9)^2} = -\frac{x^2 + 9}{(x^2 - 9)^2}$$

$f'(x) = 0 \Leftrightarrow x^2 + 9 = 0$. Thus no critical points in domain.



3. $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{x^2 + 9}{(x^2 - 9)^2}\right)$

$$= \frac{-2x(x^2 - 9)^2 + (x^2 + 9)2(x^2 - 9)2x}{(x^2 - 9)^4}$$

$$= \frac{-2x(x^2 - 9) + (x^2 + 9)2 \cdot 2x}{(x^2 - 9)^3} = \frac{2x(x^2 + 27)}{(x^2 - 9)^3}$$

$f''(x) = 0 \Leftrightarrow x = 0.$

4.

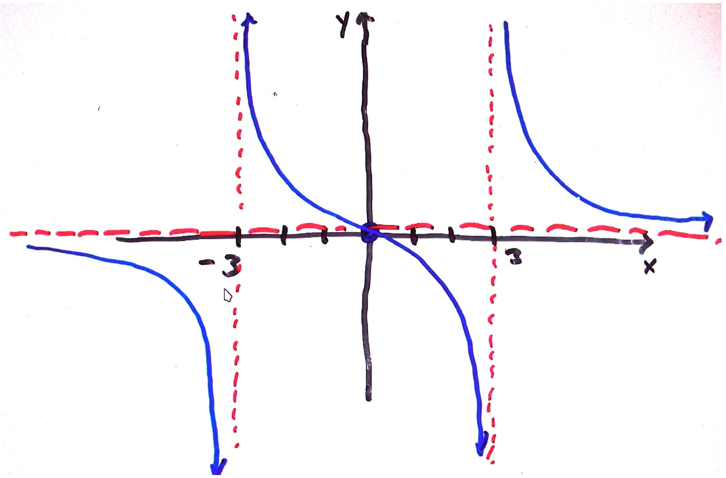
x	$f(x) = x/(x^2 - 9)$
0	0

5. There are vertical asymptotes at $x = 3$ an $x = -3$ since $\lim_{x \to \pm 3} x/(x^2 - 9) = \pm\infty$, i.e. $f(x)$ has an "infinite limit".

$$\lim_{x \to \pm\infty} \frac{x}{x^2 - 9} = \lim_{x \to \pm\infty} \frac{x}{x^2} = \lim_{x \to \pm\infty} \frac{1}{x} = 0.$$

Horizontal asymptote $y = 0$.

6.



7. Questions?