Calculus I - Lecture 16
Minima and Maxima & Mean Value Theorem

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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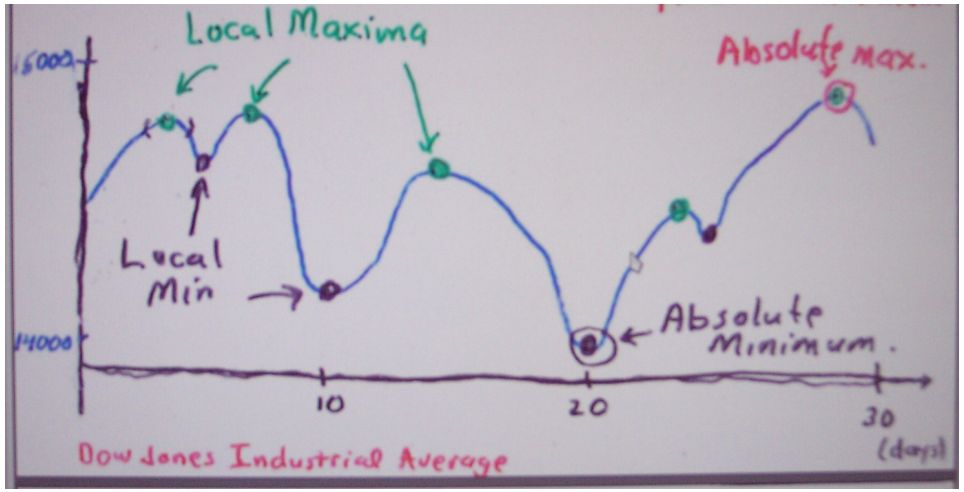
Extremal Values of Function

One of the most important applications of calculus is optimization of functions

Extrema can be divided in the following subclasses:

- Maxima and Minima
- Absolute (or global) and local (or relative) Extrema

Extrema, Maxima and Minima are the plural form of Extremum, Maximum and Minimum, respectively.



Definition (Absolute Extrema)

Let $f(x)$ be a function defined on on interval I and let $a \in I$.

1. *We say that $f(x)$ has an* **absolute maximum** *at $x = a$ if $f(a)$ is the maximal value of $f(x)$ on I. That is*

$$f(a) \geq f(x) \text{ for all } x \in I.$$

2. *We say that $f(x)$ has an* **absolute minimum** *at $x = a$ if $f(a)$ is the minimal value of $f(x)$ on I. That is*

$$f(a) \leq f(x) \text{ for all } x \in I.$$

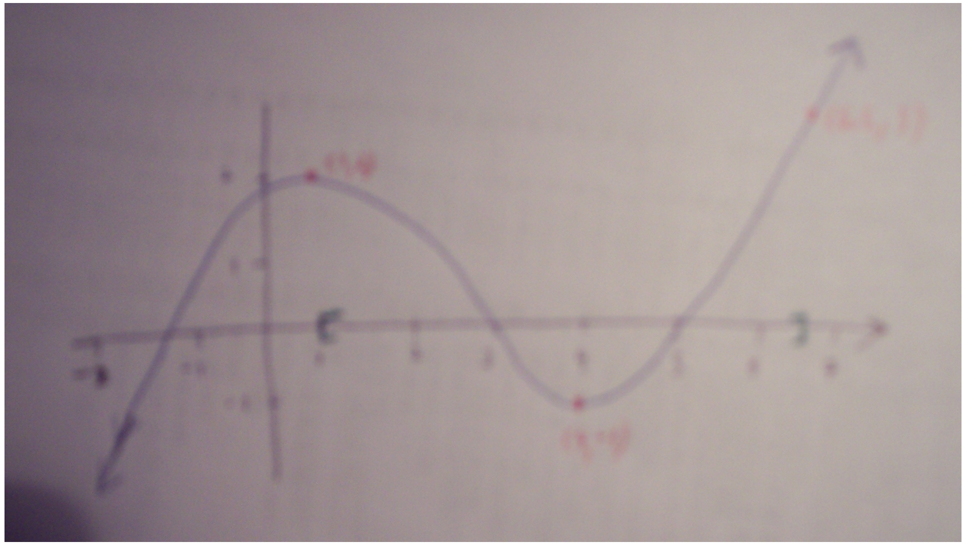
Definition (Local Extrema)

Let $f(x)$ be a function.

1. *We say that $f(x)$ has an* **local maximum** *at $x = a$ if $f(a)$ is the maximal value of $f(x)$ on some open interval I inside the domain of f containing a.*
2. *We say that $f(x)$ has an* **local minimum** *at $x = a$ if $f(a)$ is the minimal value of $f(x)$ on some open interval I inside the domain of f containing a.*

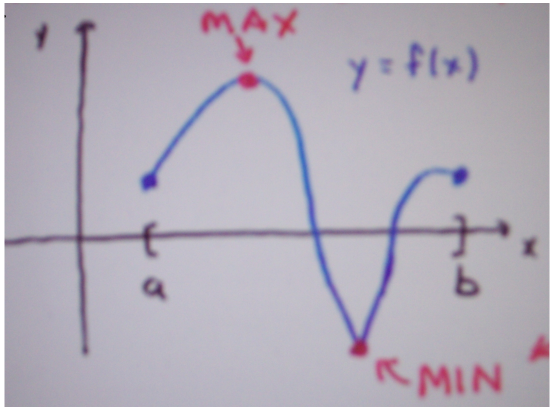
In the above situation the **value** $f(a)$ is called a global (or local) maximum (or minimum).

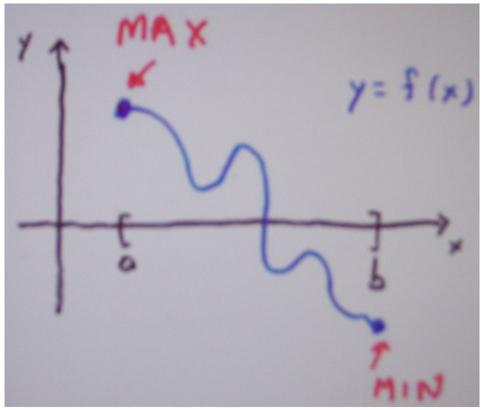
Example:



In this section out interest is in finding the (absolute) maximal and minimal values of a function on a closed interval $[a, b]$.

Where can this occur?

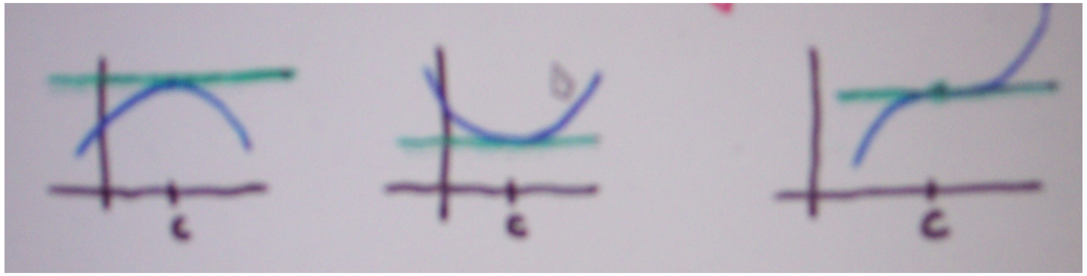




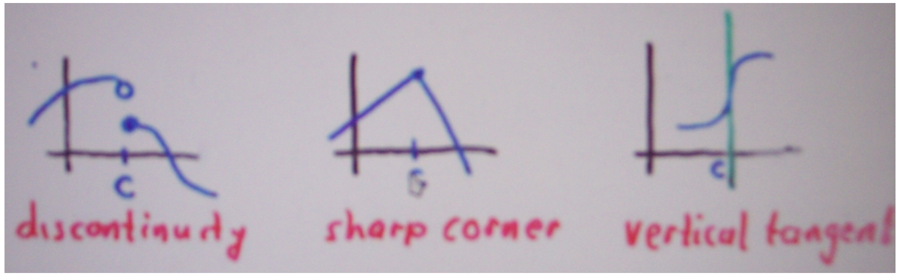
Definition

A point c in the domain of a function $f(x)$ is called a critical point if either

1. $f'(c) = 0$ *(horizontal tangent)*

2. *or $f'(c)$ does* **not** *exist.*

Theorem

If $f(c)$ is a local maximum or minimum, then c is a critical point of $f(x)$.

Note: The converse does **not** hold, i.e., if $f'(c) = 0$ then $f(c)$ is not necessarily a maximum or minimum.

Example: Find the local minima and maxima of $f(x) = x^3$.

Solution:

By the theorem, we have to find the critical points.

Since $f'(x) = 3x^2$, which is defined everywhere, the critical points occur where $f'(x) = 0$. From $f'(x) = 3x^2 = 0$ we find $x = 0$ as the only critical point.

Since for all $x < 0$ one has $f(x) < 0$ and for $x > 0$ one has $f(x) > 0$ we see that $f(0) = 0$ is **not** a local extremum.

The function $f(x) = x^3$ has no local minima or maxima.

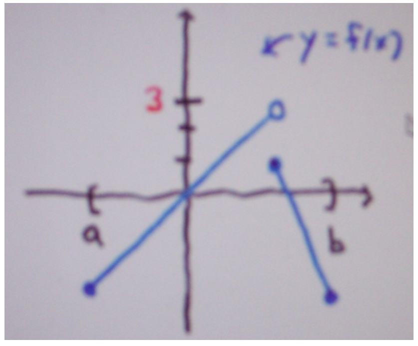
Theorem

Suppose that $f(x)$ is continuous on the closed interval $[a, b]$. Then $f(x)$ attains its absolute maximum and minimum values on $[a, b]$ at either:

- *A critical point*
- *or one of the end points a or b.*

Note: If $f(x)$ is **not** continuous on $[a, b]$ then this theorem fails.

Example: What is the maximum value of $f(x)$ on the interval $[a, b]$?



Solution: There is none.

Example:

a) Find the critical points for the function $f(x) = 3x - x^3$.

b) Find the maximal and minimal values of $f(x) = 3x - x^3$ on the interval $[-1, 3]$.

Solution:

a) $f'(x) = 3 - 3x^2 = 3(1 - x^2)$

$f'(x) = 0$ when $x^2 = 1$ or $x = \pm 1$.

$f'(x)$ exists for all real numbers.

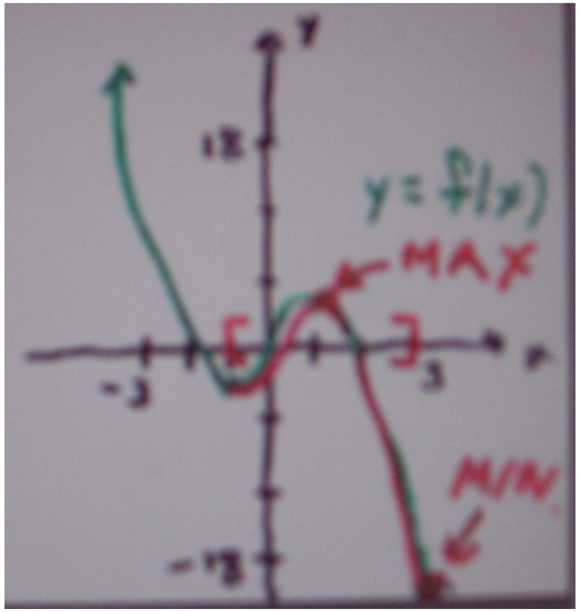
The only critical points are ± 1.

b) We make a table with the critical points inside the interval and its endpoints.

x	$3x - x^3$
1	$3 - 1 = 2$
-1	$-3 - (-1) = -2$
3	$3 \cdot 3 - 3^3 = -18$

Maximal value is 2 at $x = 1$,

Minimal value is -18 at $x = 3$.


y = f(x)
MAX
MIN
-3
3

Example: Find the minimal and maximal value of $f(x) = x(2-x)^{1/3}$ on the interval $[1,3]$.

Solution: Step 1: Find the critical points.

$$f'(x) = 1\cdot(2-x)^{1/3} + x\cdot\tfrac{1}{3}(2-x)^{-2/3}(-1)$$
$$= \frac{2-x}{(2-x)^{2/3}} - \frac{x}{3(2-x)^{2/3}}$$
$$= \frac{3(2-x)-x}{3(2-x)^{2/3}} = \frac{6-4x}{3(2-x)^{2/3}}$$

$f'(x) = 0 \Rightarrow 6-4x = 0 \Rightarrow 4x = 6 \Rightarrow x = \frac{3}{2}$

$f'(x)$ does not exist $\Rightarrow 2-x = 0 \Rightarrow x = 2$

Step 2: Make table with critical points & endpoints

x	$x(2-x)^{1/3}$
$3/2$	$\frac{3}{2}(\frac{1}{2})^{1/3} \approx 1.19$
2	$2\cdot 0^{1/3} = 0$
1	$1\cdot 1^{1/3} = 1$
3	$3(-1)^{1/3} = -3$

Maximal value is $\frac{3}{2\sqrt[3]{2}}$, minimal value is -3.

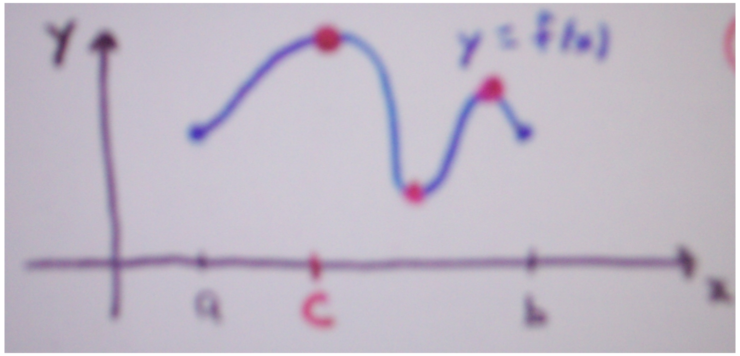
Mean Value Theorem

We start with an important special case of the Mean Value Theorem (MVT)

Theorem (Rolle's Theorem)

Suppose $f(x)$ is a continuous function on $[a, b]$, is differentiable on (a, b), and $f(a) = f(b)$.
Then there exists a c in (a, b) with $f'(c) = 0$.

Note: There can be more than one such value of c.



Proof: Follows from the results of the last section!

Example: Let $f(x) = x^2 - 3x$. Show that all hypothesis for Rolle's Theorem are satisfied for the interval $[1, 2]$ and find all c as in the theorem.

Solution:
Since $f(x)$ is a polynomial it is continuous on $[1, 2]$ and the derivative $f'(x)$ exists on $(1, 2)$.

$$f(1) = 1^2 - 3 \cdot 1 = -2, \qquad f(2) = 2^2 - 3 \cdot 2 = -2$$

Thus $f(1) = f(2)$.

Find c:

$f'(x) = 2x - 3 = 0 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$.

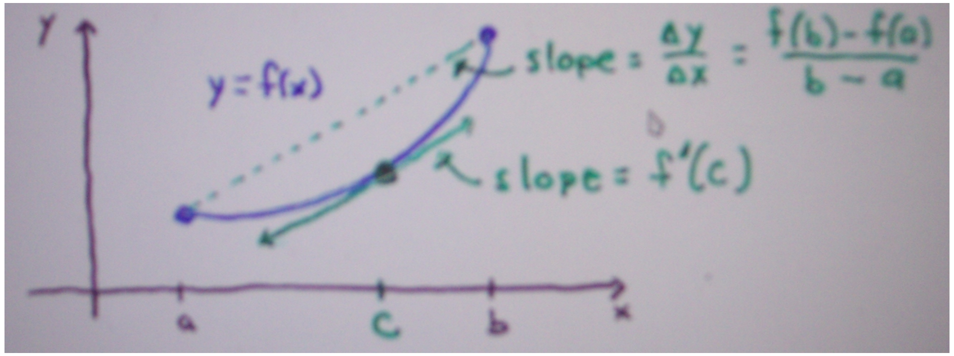
$c = \frac{3}{2}$ satisfies the conclusion of Rolle's Theorem.

Theorem (Mean Value Theorem)

Suppose $f(x)$ is a continuous function on $[a, b]$ and is differentiable on (a, b).

Then there exists a c in (a, b) with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Note: Again, there can be more than one such value of c.



Intuitive version: There is always a time when your instantaneous velocity equals your average velocity.

Example: Let $f(x) = x^3$. Find all c in $[-1, 3]$ satisfying the conclusion of the MVT.

Solution:

First note that $f(x)$ is continuous on $[-1, 3]$ and differentiable on $(-1, 3)$.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(-1)}{3 - (-1)} = \frac{27 - (-1)}{4} = \frac{28}{4} = 7.$$

$$f'(x) = 3x^2$$

MVT: $7 = 3c^2 \Rightarrow c^2 = \frac{7}{3} \Rightarrow c = \pm\sqrt{\frac{7}{3}}$

Only $c = \sqrt{\frac{7}{3}}$ is in $(-1, 3)$.

$c = \sqrt{\frac{7}{3}}$ satisfies the conclusion of the MVT.