Calculus I - Lecture 14 - Related Rates

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

March 10, 2014

Problem 8 of Exam 2

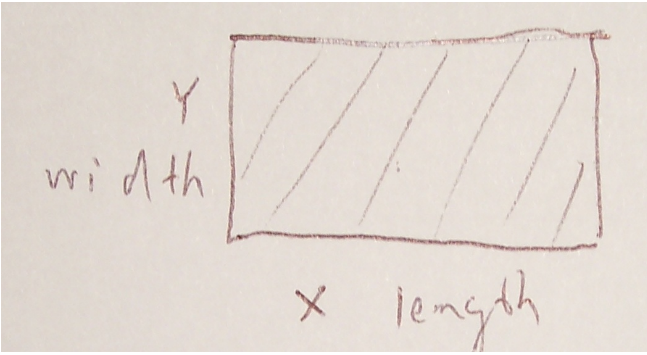
Find the derivative, simplify, and determine where it is zero.
$y = \ln(3 + e^{\cos(5x)})$.

Solution with Mathematica

```
Mathematica 7.0 for Sun Solaris SPARC (64-bit)
Copyright 1988-2008 Wolfram Research, Inc.

In[1]:= y=Log[3+E^Cos[5x]]

                 Cos[5 x]
Out[1]= Log[3 + E        ]

In[2]:= D[y,x]

            Cos[5 x]
        -5 E         Sin[5 x]
Out[2]= ---------------------
                 Cos[5 x]
            3 + E

In[3]:= Reduce[%==0,x]

                                                 2 Pi C[1]          Pi + 2 Pi C[1]
Out[3]= C[1] \[Element] Integers && (x == --------- || x == --------------)
                                                     5                    5
```

Example: A rectangle is changing in such a manner that its length is increasing 5 ft/sec and its width is decreasing 2 ft/sec. At what rate is the area changing at the instant when the length equals 10 feet and the width equals 8 feet?

Solution:



A area of triangle

Given: $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ ft/sec $\frac{\mathrm{d}y}{\mathrm{d}t} = -2$ ft/sec

Find: $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $x = 100$ ft.

$A = xy$ and so $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}(xy) = \dfrac{\mathrm{d}x}{\mathrm{d}t} \cdot y + x \cdot \dfrac{\mathrm{d}y}{\mathrm{d}t}$

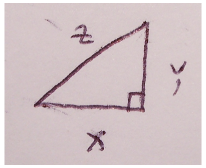
$\dfrac{\mathrm{d}A}{\mathrm{d}t} = 5 \cdot 8 + 10 \cdot (-2) = 40 - 20 = 20.$

The area is increasing by a rate of 20 $\mathrm{ft}^2/\mathrm{sec}$.

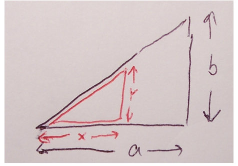
Related Rates — Main steps

1. Draw a picture and label variables.
2. State the problem mathematically:
   Given . . . , Find . . . .
3. Find a relationship between the variables:
   a) Pythagorean Theorem
   b) Similar triangles
   c) Volume/Area formulas
   d) Trigonometric Relations
4. Take implicit derivatives $\frac{\mathrm{d}}{\mathrm{d}t}$ and solve for the asked quantity.
5. Find the the remaining variables at that instance.
6. Plug in the specific values for the variables.
7. State the answer in a complete sentence with the correct units.

Possible relationships between variables

1) Pythagorean Theorem 

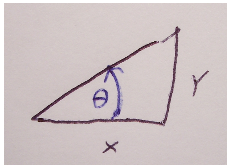
$x^2 + y^2 = z^2$

2) Similar Triangles 

$\frac{y}{b} = \frac{x}{a}$ or $\frac{y}{x} = \frac{b}{a}, \ldots$

3) Volume/Area formulas

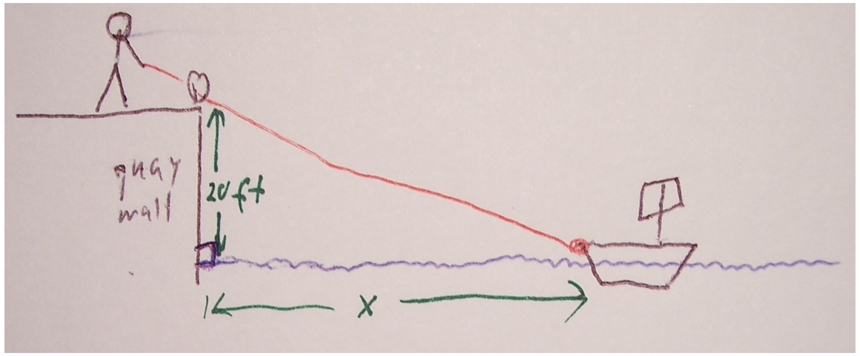
Area rectangle $A = xy$, Area circle $A = \pi r^2$,
Volume sphere $V = \frac{4}{3}\pi r^3$, Surface area sphere $A = 4\pi r^2$, etc.

4) Trigonometric Relations 

$\frac{y}{x} = \tan\theta$

Example: (Pythagorean type) A boat is pulled on shore by rope from a 20 feet high quay wall. If the rope is pulled at a rate of 15 ft/min, at what rate is the boat approaching the shore when it is 100 ft away from the shore?

Solution:

1) 

2) Given: $\dfrac{\mathrm{d}z}{\mathrm{d}t} = -15$ ft/min

Find: $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 100$ ft.

3) $z^2 = x^2 + 20^2$ (Pythagorean Theorem)

4) $$\frac{\mathrm{d}}{\mathrm{d}t} z^2 = \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + 20^2)$$

$$2z \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2z}{2x} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{z}{x} \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

5) We need also z.

If $x = 100$ then $z^2 = 100^2 + 20^2 = 10400$.

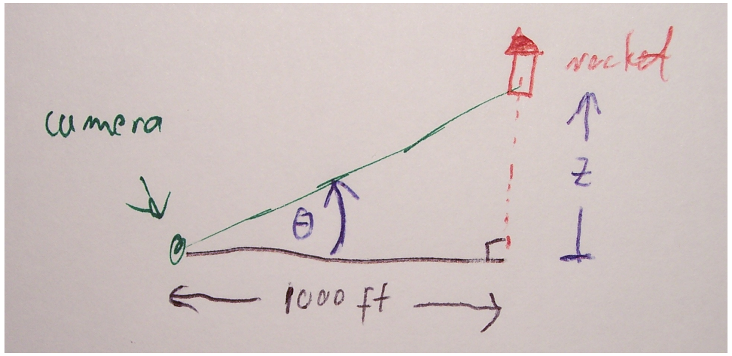
Thus $z = \sqrt{10400} = 10\sqrt{104} = 20\sqrt{26}$

6) $$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=100} = \frac{20\sqrt{26}}{100} \cdot (-15) = \frac{\sqrt{26}}{5} = -3\sqrt{26}$$

7) The boat approaches the shore by the rate $3\sqrt{26}$ ft/min.

Example: (Trigonometric type) A camera on ground tracks a rocket start 1000 feet away. At a given instant, the camera is pointing at an angle of 45° upward which is growing at a rate of 20° per second. How fast is the rocket traveling at this instant?

Solution:

1) 

2) Given: $\dfrac{d\theta}{dt} = 20 \text{ deg/sec} \cdot \dfrac{\pi}{180} \text{ rad/deg} = \dfrac{\pi}{9} \text{ rad/sec}$

Find: $\dfrac{dz}{dt}$ when $\theta = 45^\circ = \frac{\pi}{4}$.

3) $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{z}{1000}$

4) $\dfrac{\mathrm{d}}{\mathrm{d}t} \tan\theta = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{z}{1000}$

$$\sec^2\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1000}\frac{\mathrm{d}z}{\mathrm{d}t}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 1000 \cdot \sec^2\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1000 \cdot \frac{1}{\cos^2\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

5) —

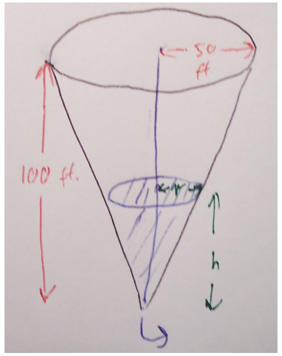
6) $\left.\dfrac{\mathrm{d}z}{\mathrm{d}t}\right|_{\theta=\frac{\pi}{4}} = 1000 \cdot \dfrac{1}{\cos^2(\pi/4)} \cdot \dfrac{\pi}{9}$

$$= 1000 \cdot \frac{1}{(1/\sqrt{2})^2} \cdot \frac{\pi}{9} = \frac{2000\pi}{9}$$

7) The rocket is ascending with a velocity of $\dfrac{2000\pi}{9}$ ft/sec.

Example: (Volume type) A cusp downward pointing conical tank has a hight of 100 feet and a radius of 50 feet. The hight of water in the tank is falling at a rate of 2 feet per hour. How fast is the tank losing water when the water hight is 10 feet.

Solution:

1) 

V Volume of water

2) Given: $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -2$ ft/hour

Find: $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ when $h = 10$ ft.

3) $V = \frac{1}{3}\pi r^2 h$ (Volume of cone)

$\dfrac{r}{h} = \dfrac{50}{100} = \dfrac{1}{2}$ (similar triangles)

4) $$\frac{\mathrm{d}}{\mathrm{d}t}V = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{3}\pi r^2 h)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V = \frac{1}{3}\pi\left(2r\cdot\frac{\mathrm{d}r}{\mathrm{d}t}\cdot h + r^2\cdot\frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

5) Need r and $\frac{\mathrm{d}r}{\mathrm{d}t}$.

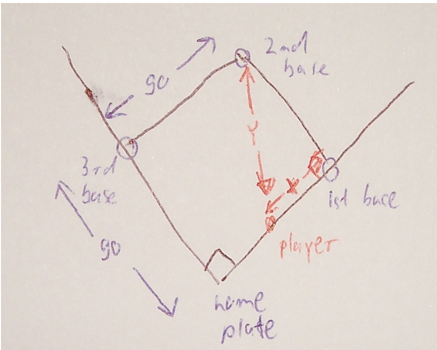
$\frac{r}{h} = \frac{50}{100} = \frac{1}{2}$ gives $r = \frac{1}{2}h = \frac{10}{2} = 5$ and
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-2}{2} = -1.$$

6) $$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{h=10} = \frac{1}{3}\pi\left(2\cdot 5\cdot(-1)\cdot 10 + 5^2\cdot(-2)\right)$$
$$= \frac{1}{3}\pi\cdot(-100-50) = -50\pi$$

7) The water is flowing out of the tank with $50\pi\ \mathrm{ft}^3/\mathrm{hr}$.

Example: Recall that in baseball the home plate and the three bases form a square of side length 90 ft. A batter hits the ball and runs to the first base at 24 ft/sec. At what rate is his distance from the 2nd base decreasing when he is halfway to the first base.

Solution:

1) 

x distance between player and first base.

y distance between player and 2nd base.

2) Given: $\frac{\mathrm{d}x}{\mathrm{d}t} = -24$ ft/sec.

Find: $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x = \frac{90}{2} = 45$ ft.

3) $$y^2 = x^2 + 90^2$$

4) $$\frac{\mathrm{d}}{\mathrm{d}t}(y^2) = \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + 90^2)$$

$$2y \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{x}{y} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

5) Need also y.

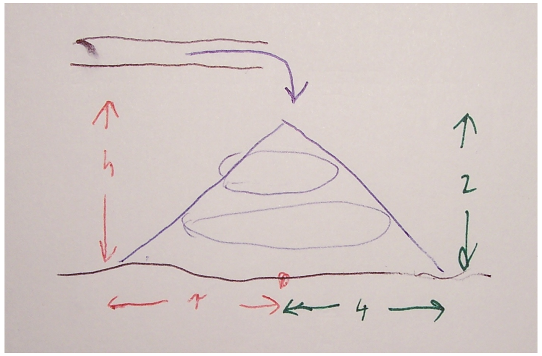
$$x = 45, \qquad y = \sqrt{x^2 + 90^2} = \sqrt{45^2 + 90^2} = 45\sqrt{5}$$

6) $$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{45}{45\sqrt{5}} \cdot (-24) = -\frac{24}{\sqrt{5}}$$

7) The distance between the player and the 2nd base decreases by a rate of $\frac{24}{\sqrt{5}}$ ft/sec.

Example: Grain flows into a conical pile such that the height increases 2 ft/min while the radius increases 3 ft/min. At what rate is the volume increasing when the pile is 2 feet high and has a radius of 4 feet.

Solution:

1) 

2) $\frac{\mathrm{d}h}{\mathrm{d}t} = 2$ ft/min when $h = 2$ ft

$\frac{\mathrm{d}r}{\mathrm{d}t} = 3$ ft/min when $r = 4$ ft

3) Volume of the cone $V = \frac{1}{3}\pi r^2 h$

4) $$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\pi\left[2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot h + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right]$$

5) —

6) Evaluating at $h = 2$ and $r = 4$ gives:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left[2 \cdot 4 \cdot 3 \cdot 2 + 4^2 \cdot 2\right] = \frac{1}{3}\pi\,[48 + 82]$$
$$= \frac{80\pi}{3}$$

7) The volume is increasing at a rate of $\frac{80\pi}{3}$ $\mathrm{ft}^3/\mathrm{min}$.