Calculus I - Lecture 13 - Review Exam 2

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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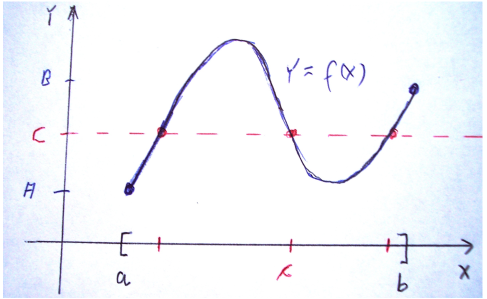
Theorem (Intermediate Value Theorem (IVT))

Let $f(x)$ be **continuous** *on the interval $[a, b]$ with $f(a) = A$ and $f(b) = B$.*

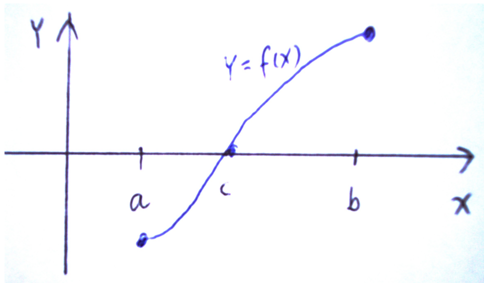


Given any value C between A and B, there is at least one point $c \in [a, b]$ with $f(c) = C$.

Important special case of the IVT:

Suppose that $f(x)$ is **continuous** on the interval $[a, b]$ with $f(a) < 0$ and $f(b) > 0$.

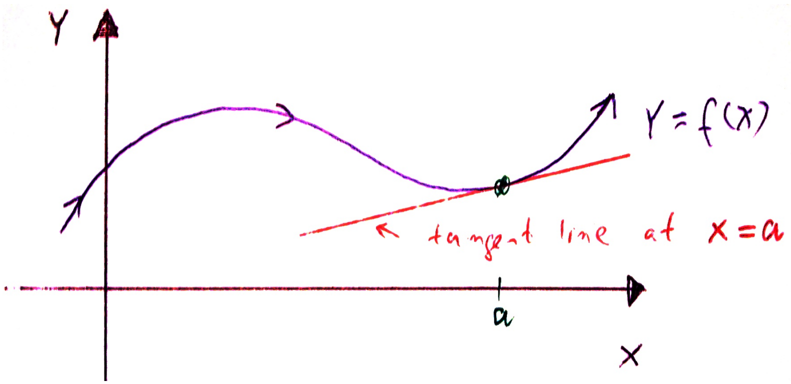
Then there is a point $c \in [a, b]$ where $f(c) = 0$.



Geometric View of the Derivative

Recall, the slope of a line is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$



Definition (Tangent Line)

A tangent line is a line that (in general)

1. *touches the graph at one point (near that point) and*
2. *has a slope equal to the slope of the curve.*

If the curve is a line segment, the tangent line coincides with the segment.

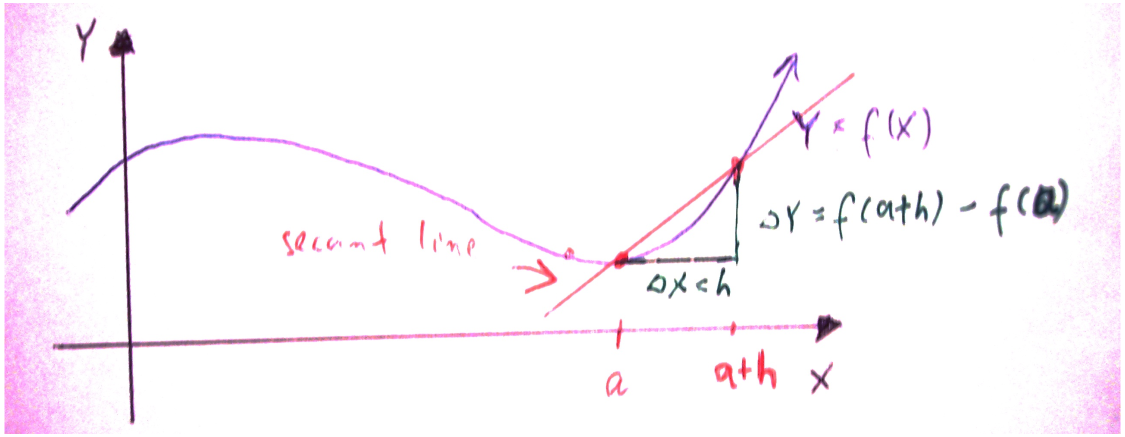
Slope of a curve at $x = a$ equals $m_{\tan}$ = slope of tangent line.

Definition (Derivative — geometric)

The **derivative** *of a function $f(x)$ at $x = a$, denoted $f'(a)$ (pronounced "f prime of a"), is the slope of the curve $y = f(x)$ at $x = a$.*

$$\begin{aligned} f'(a) &= \text{the derivative of f(x) at } a \\ &= m_{\tan}, \text{ the slope of the tangent line.} \end{aligned}$$

Algebraic View of the Derivative



Let us determine the slope of the curve at $x = a$.

Let $h =$ tiny positive number (e.g. 0.0001)

$m_{\text{sec}} =$ slope of the secant line shown above

$$= \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

$$m_{\tan} = \lim_{h \to 0} m_{\text{sec}}$$

Definition (Derivative — algebraic)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Important Derivatives

$f(x)$	$f'(x)$	
c	0	(c any real constant)
x	1	
x^n	$n\,x^{n-1}$	(n any real constant)
e^x	e^x	
b^x	$(\ln b)\, b^x$	(b any positive constant)
$\ln x$	$\frac{1}{x}$	
$\log_b x$	$\frac{1}{\ln b}\,\frac{1}{x}$	(b any positive constant)
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\sec x$	$\sec x \tan x$	
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	
$\arctan x$	$\frac{1}{1+x^2}$	

Important Rules

Sum and Difference Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \pm g(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

Constant Factor Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c \cdot f(x)\right) = c \frac{\mathrm{d}}{\mathrm{d}x} f(x) \qquad (c \text{ a constant})$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot g(x) + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot g(x) - f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} g(x)}{g(x)^2}$$

Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Inverse Function Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Example: Find the derivatives.

a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, 2^{x^2} = (\ln 2) 2^{x^2} \cdot 2x = 2(\ln 2) x\, 2^{x^2}$

b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, e^x \arcsin(x^2) = e^x \arcsin(x^2) + e^x \cdot \dfrac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$$= \left(\arcsin(x^2) + \frac{2x}{\sqrt{1-x^4}} \right) e^x$$

c) $\tan^3(1-x^2) = 3\tan^2(1-x^2) \cdot \sec^2(1-x^2)(-2x)$

$$= -6x \left(\tan(1-x^2) \sec(1-x^2) \right)^2$$

d) $\dfrac{\arctan x}{1+\ln x} = \dfrac{\frac{1}{1+x^2}(1+\ln x) - \arctan(x)\frac{1}{x}}{(1+\ln x)^2}$

$$= \frac{1}{(1+x^2)(1+\ln x)} - \frac{\arctan x}{x(1+\ln x)^2}$$

Example: Use logarithmic differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for

$$y = x^x(x^2+1)^{5/2}.$$

Solution:

$$\ln y = \ln(x^x) + \ln((x^2+1)^{5/2})$$

$$\ln y = x\ln x + \frac{5}{2}\ln(x^2+1)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln y = \frac{\mathrm{d}}{\mathrm{d}x}(x\ln x) + \frac{\mathrm{d}}{\mathrm{d}x}(\frac{5}{2}\ln(x^2+1))$$

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = (1\cdot\ln x + x\cdot\frac{1}{x}) + \frac{5}{2}\frac{1}{x^2+1}\cdot 2x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^x(x^2+1)^{5/2}\left[1+\ln x+\frac{5x}{x^2+1}\right]$$

Example: Find the tangent line to the curve $x^2y^3 = y^2 + 3$ at $(2, 1)$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2 + 3)$$

$$2xy^3 + x^2 \cdot 3y^2 \cdot y' = 2y \cdot y'$$

$$(3x^2y^2 - 2y) \cdot y' = -2xy^3$$

$$y' = \frac{-2xy^3}{3x^2y^2 - 2y} = -\frac{2xy^2}{3x^2y - 2}$$

$$y'\Big|_{(2,1)} = -\frac{2 \cdot 2 \cdot 1^2}{3 \cdot 2^2 \cdot 1 - 2} = -\frac{4}{10} = -\frac{2}{5}$$

Tangent line:

$$y = m(x - x_0) + y_0$$

$$y = -\frac{2}{5}(x - 2) + 1 = -\frac{2}{5}x + \frac{9}{5}$$

Example: Recall that in baseball the home plate and the three bases form a square of side length 90 ft. A batter hits the ball and runs to the first base at 24 ft/sec. At what rate is his distance from the 2nd base decreasing when he is halfway to the first base.

Solution:

Let x distance between player and first base.

Let y distance between player and 2nd base.

Given $\frac{dx}{dt} = -24$ ft/sec. Find $\frac{dy}{dt}$ when $x = \frac{90}{2} = 45$ ft.

$$y^2 = x^2 + 90^2$$

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(x^2 + 90^2)$$

$$2y \cdot \frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$$

$$x = 45, \qquad y = \sqrt{x^2 + 90^2} = \sqrt{45^2 + 90^2} = 45\sqrt{5}$$

Thus $$\frac{dy}{dt} = \frac{45}{45\sqrt{5}} \cdot (-24) = -\frac{24}{\sqrt{5}} \text{ ft/sec.}$$

Example: Grain flows into a conical pile such that the height increases 2 ft/min while the radius increases 3 ft/min. At what rate is the volume increasing when the pile is 2 feet high and has a radius of 4 feet.

Solution:

Volume of the cone: $V = \frac{1}{3}\pi r^2 h$

$\frac{dh}{dt} = 2$ ft/min when $h = 2$

$\frac{dr}{dt} = 3$ ft/min when $r = 4$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\pi\left[2r \cdot \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt}\right]$$

Evaluating at $h = 2$ and $r = 4$ gives:

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2 \cdot 4 \cdot 3 \cdot 2 + 4^2 \cdot 2\right] = \frac{1}{3}\pi\,[48 + 82]$$

$$= \frac{80\pi}{3} \text{ ft}^3/\text{sec}.$$

Example: Does the function $f(x) = 2x^3 - \sin(x-1)$ has a zero?

Solution:

We have:

$$f(-2) = 2(-2)^3 - \sin(-3) \leq -16 + 1 = -15 < 0$$

$$f(2) = 2 \cdot 2^3 - \sin(1) \geq 16 - 1 = 15 > 0$$

Since $f(x)$ is a continuous function, it has by the intermediate value theorem a zero on the interval $[-2, 2]$.