Calculus I - Lecture 12 - Implicit Differentiation

Lecture Notes:

http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:

http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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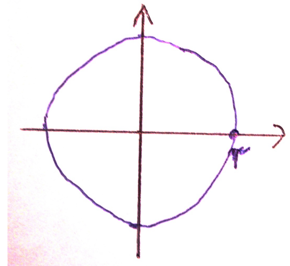
Implicit Differentiation

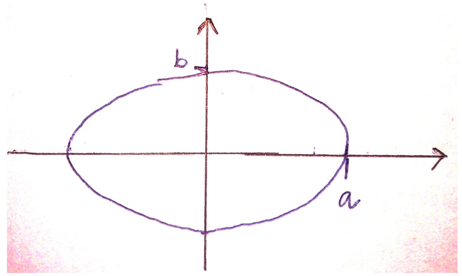
Implicit differentiation is a method for finding the slope of a curve, when the equation of the curve is not given in "explicit" form $y = f(x)$, but in "implicit" form by an equation $g(x, y) = 0$.

Examples

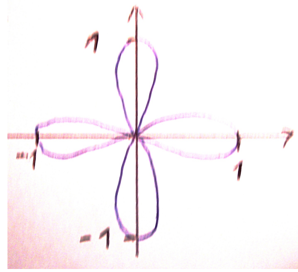
1) Circle $x^2 + y^2 = r$



2) Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



3) 4-leaf clover $(x^2 + y^2)^3 = (x^2 - y^2)^2$



Example: a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation given that $x^2 + y^2 = 25$.

General Procedure

1. **Take** $\dfrac{\mathrm{d}}{\mathrm{d}x}$ of both sides of the equation.
2. Write $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ and **solve for** y'.

Solution:

Step 1

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}25$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}y^2 = 0$$

Use: $\dfrac{\mathrm{d}}{\mathrm{d}x}y^2 = \dfrac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\big)^2 = 2f(x) \cdot f'(x) = 2y \cdot y'$

$$2x + 2y \cdot y' = 0$$

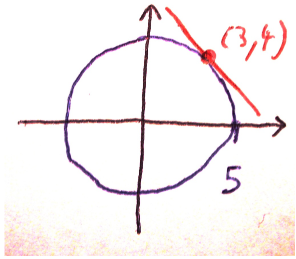
Step 2

$$2y \cdot y' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

Example:

b) What is the slope of the circle at $(3, 4)$?



c) What is the slope at $(5, 0)$

Solution:

b) The slope is $y'\Big|_{(3,4)} = -\frac{3}{4}$

c) The slope is $y'\Big|_{(5,0)} = -\frac{5}{0}$ (undefined)

The curve has a vertical tangent at $(5, 0)$.

Restated derivative rules using y, y' notation

Let $y = f(x)$ and $y' = f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$.

General Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} y^n = ny^{n-1} \cdot y'$$

Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} g(y) = g'(y) \cdot y'$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} h(x) \cdot g(y) = h'(x) \cdot g(y) + h(x) \cdot g'(y) \cdot y'$$

$$\frac{\mathrm{d}}{\mathrm{d}x} h(y) \cdot g(y) = h'(y) \cdot y' \cdot g(y) + h(y) \cdot g'(y) \cdot y'$$

Exponential Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{g(y)} = e^{g(y)} \cdot g'(y) \cdot y'$$

Example: Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation for the curve $xy^2 = \sin\big(y^3\big)$

Solution: Two steps.

Step 1 (take $\dfrac{\mathrm{d}}{\mathrm{d}x}$ of both sides)

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy^2) = \frac{\mathrm{d}}{\mathrm{d}x}\sin\big(y^3\big)$$

product rule chain rule

$$(\frac{\mathrm{d}}{\mathrm{d}x}x)\cdot y^2 + x\cdot(\frac{\mathrm{d}}{\mathrm{d}x}y^2) = \cos(y^3)\cdot\frac{\mathrm{d}}{\mathrm{d}x}y^3$$

$$1\cdot y^2 + x\cdot 2y\cdot y' = \cos(y^3)\cdot 3y^2\cdot y'$$

Step 2 (solve for y')

$$2xyy' - 3\cos(y^3)y^2y' = -y^2$$

$$\big(2xy - 3\cos(y^3)y^2\big)y' = -y^2$$

$$y' = \frac{-y^2}{2xy - 3\cos(y^3)y^2} = \frac{y}{3y\cos(y^3) - 2x}$$

Make sure that there is no y' left on right-hand side.

Example: Find the equation of the tangent line to the curve $x^2y - y^3 = x - 7$ at $(1, 2)$.

Solution: We first find y'.

Step 1

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y - y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(x - 7)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \cdot y + x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x} y - \frac{\mathrm{d}}{\mathrm{d}x}(y^3) = 1 - 0$$

$$2xy + x^2 \cdot y' - 3y^2 \cdot y' = 1$$

Step 2

$$(x^2 - 3y^2) \cdot y' = 1 - 2xy$$

$$y' = \frac{1 - 2xy}{x^2 - 3y^2}$$

$$y'\Big|_{(1,2)} = \frac{1 - 2 \cdot 1 \cdot 2}{1^2 - 3 \cdot 2^2} = \frac{-3}{-11} = \frac{3}{11}$$

Point slope form of tangent line: $y = m(x - x_1) + y_1$

$$y = \tfrac{3}{11}(x - 1) + 2 = \tfrac{3}{11}x + \tfrac{19}{11}$$

Does this work for every implicit given curve $g(x, y) = 0$?

Note that every equation in x and y can be written in this form by bringing everything on the left-hand side.

Answer is **yes**!

Solution:

Step 1

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x, y) = \frac{\mathrm{d}}{\mathrm{d}x} 0$$

$$g_x(x, y) + g_y(x, y) \cdot y' = 0$$

Here, g_x and g_y are the "partial derivatives" of $g(x, y)$ with respect to the first variable x resp. the second variable y (ignoring here that y depends on x). That is, one differentiates $g(x, y)$ for x and keeps y fixed and vice versa.

Step 2

$$y' = -\frac{g_x(x, y)}{g_y(x, y)}.$$

Example: Find the coordinates of the four points on the *lemniscate curve* $(x^2+y^2)^2 = 4(x^2-y^2)$ on which the tangent line is horizontal.

Solution: We compute y' by implicit differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)^2 = \frac{\mathrm{d}}{\mathrm{d}x}(4(x^2-y^2))$$

$$2(x^2+y^2)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2) = 8x - 4\frac{\mathrm{d}}{\mathrm{d}x}(y^2)$$

$$2(x^2+y^2)(2x+2y\cdot y') = 8x - 8y\cdot y'$$

$$4x(x^2+y^2) - 8x = -8y\cdot y' - 4(x^2+y^2)yy'$$

$$y' = \frac{4x(x^2+y^2)-8x}{-8y-4(x^2+y^2)y} = -\frac{x(x^2+y^2)-2x}{y(x^2+y^2)+2y}$$

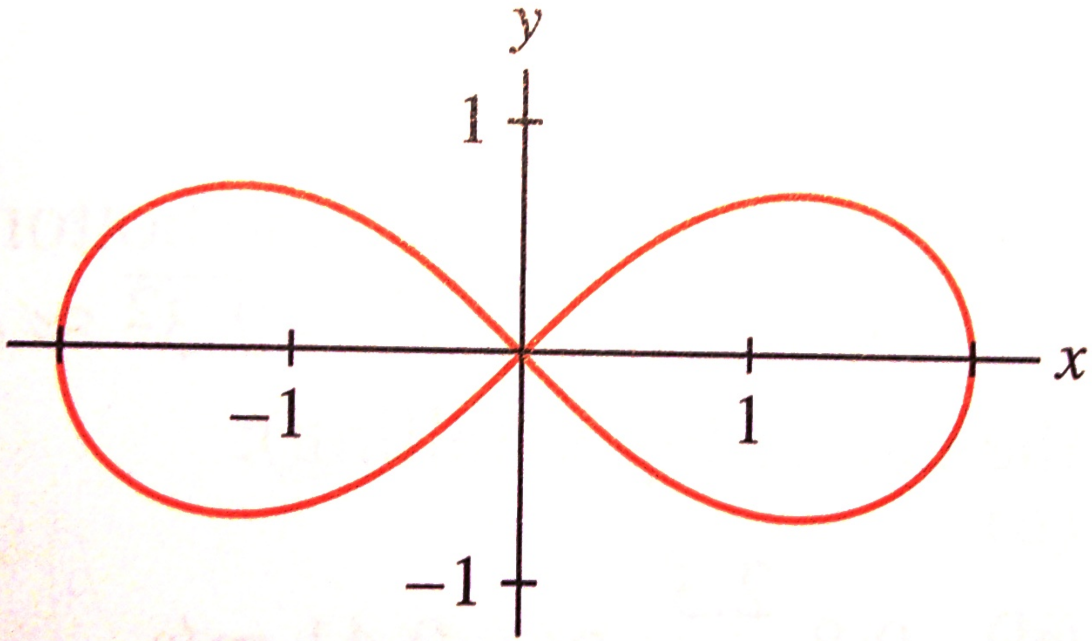
We have to solve for $y'=0$ which requires:

$x(x^2+y^2)-2x=0$, i.e. $x=0$ or $x^2+y^2=2$.

For $x=0$ one needs $y^4=-4y^2$, i.e. $y=0$ is only solution.

For $x^2+y^2=2$ one has $2^2=4(x^2-y^2)$ which gives $2x^2=3$ or $x=\pm\sqrt{3/2}$ and $y=\pm\sqrt{1/2}$.

Only the four points $(\pm\sqrt{3/2},\pm\sqrt{1/2})$ but not $(0,0)$ are solutions.



Lemniscate curve: $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

Example: Find the first and second derivative of $y = f(x) = x^x$.

Solution: Note that $f(x)$ is **not** of the form b^x or x^b and thus the corresponding rules **don't apply**.

a) We use logarithmic differentation:

$$\ln y = \ln(x^x) = x \ln x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln y = \ln(x^x) = \frac{\mathrm{d}}{\mathrm{d}x} (x \ln x)$$

$$\frac{1}{y} \cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$y' = y \cdot (\ln x + 1) = x^x(\ln x + 1)$$

b) $$y'' = \frac{\mathrm{d}}{\mathrm{d}x} y' = \frac{\mathrm{d}}{\mathrm{d}x} \left(x^x(\ln x + 1)\right)$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x} (x^x) \cdot (\ln x + 1) + x^x \cdot \frac{\mathrm{d}}{\mathrm{d}x} (\ln x + 1)$$

$$y'' = x^x(\ln x + 1) \cdot (\ln x + 1) + x^x \cdot \frac{1}{x} \quad \text{(using a) again)}$$

$$y'' = x^x\left((\ln x + 1)^2 + \frac{1}{x}\right)$$