Calculus I - Lecture 11 - Derivatives of General Exponential and Inverse Functions

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

February 26, 2014

Derivative of General Exponential functions

We know:

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$$

By the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{f(x)} = e^x \cdot f'(x)$$

Example: Find the derivative of 2^x.

Solution: Recall $2 = e^{\ln 2}$, so $2^x = \left(e^{\ln 2}\right)^x = e^{(\ln 2)x}$.

Thus $$\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{(\ln 2)x} = e^{(\ln 2)x} \frac{\mathrm{d}}{\mathrm{d}x} ((\ln 2)x)$$
$$= (\ln 2) \cdot 2^x.$$

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} b^x = (\ln b) \cdot b^x, \quad \textit{for any base } b > 0.$$

Example: Find $\dfrac{\mathrm{d}}{\mathrm{d}x} 7^{x^2}$.

Solution: We apply the chain rule with outer function $f(u) = 7^u$ and inner function $g(x) = x^2$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} 7^{x^2} &= (\ln 7) \cdot 7^{(x^2)} \cdot \frac{\mathrm{d}}{\mathrm{d}x} x^2 \\ &= 2 \ln 7 \cdot x \cdot 7^{x^2}\end{aligned}$$

Example: Find $\dfrac{\mathrm{d}}{\mathrm{d}x} 5^{5^x}$.

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} 5^{5^x} &= (\ln 5) \cdot 5^{5^x} \cdot \frac{\mathrm{d}}{\mathrm{d}x} 5^x \\ &= (\ln 5) \cdot 5^{5^x} \cdot (\ln 5) \cdot 5^x \\ &= \ln^2 5 \cdot 5^x \cdot 5^{5^x}\end{aligned}$$

The Derivative of the Natural Logarithm function

Example: Let $y = \ln x$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution:

We have $e^y = e^{\ln x} = x$.

We take now the derivative on both sides:

$$\frac{\mathrm{d}}{\mathrm{d}x} e^y = \frac{\mathrm{d}}{\mathrm{d}x} x$$

$$e^y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \qquad \text{(by the chain rule)}$$

Thus: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{e^y} = \dfrac{1}{x}$, since $e^y = x$.

We have shown the following rule:

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

Example: Find $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x - 5x^3)$.

Solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \ln(x - 5x^3) &= \frac{1}{x - 5x^3} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x - 5x^3) \\ &= \frac{1}{x - 5x^3} \cdot (1 - 15x^2) \\ &= \frac{1 - 15x^2}{x - 5x^3} \end{aligned}$$

Example: Find $\frac{\mathrm{d}}{\mathrm{d}x} x \ln x$.

Solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} x \ln x &= \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \ln x + x \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\ln x) \\ &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\ &= 1 + \ln x \end{aligned}$$

Example: Compute $\frac{\mathrm{d}}{\mathrm{d}x} \log_b x$ for any base $b > 0$.

Solution:

We have $\log_b x = \frac{\ln x}{\ln b}$.

Thus:

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_b x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln x}{\ln b} = \frac{1}{\ln b} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{\ln b} \cdot \frac{1}{x}. \qquad \text{(Remember)}$$

Example: Compute $\frac{\mathrm{d}}{\mathrm{d}x} \log_5(\log_5(x))$.

Solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \log_5(\log_5(x)) &= \frac{1}{\ln 5} \cdot \frac{1}{\log_5 x} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \log_5 x \\ &= \frac{1}{\ln 5} \cdot \frac{1}{\log_5 x} \cdot \frac{1}{\ln 5} \cdot \frac{1}{x} \\ &= \frac{1}{\ln^2 5 \cdot x \log_5 x} \end{aligned}$$

Logarithmic Differentiation

Let $y = f(x)$, $y' = \frac{dy}{dx} = f'(x)$.

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln y = \frac{y'}{y}$$

Indeed, by the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln y = \frac{\mathrm{d}}{\mathrm{d}x} \ln(f(x)) = \frac{1}{f(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{y'}{y}.$$

This is sometimes helpful to compute the derivative of a function which is mainly a combination of products, quotients or powers:

1. Take 'ln' of both sides and expand using the folowing rules:

- $\ln(AB) \; = \ln A + \ln B$
- $\ln(A/B) = \ln A - \ln B$
- $\ln(A^n) \; = n \ln A$

2. Take '$\frac{d}{dx}$' of both sides, using the theorem for the left side.
3. Solve for y'.

Example: Find $\frac{dy}{dx}$ by logarithmic differentiation, given $y = \sqrt[4]{x}\sin^3 x$.

Solution: Step 1:

$$\begin{aligned}\ln y &= \ln\left(\sqrt[4]{x}\sin^3 x\right)\\ &= \ln(\sqrt[4]{x}) + \ln(\sin^3 x)\\ &= \ln(x^{1/4}) + \ln((\sin x)^3)\\ &= \tfrac{1}{4}\ln x + 3\ln(\sin x)\end{aligned}$$

Step 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4}\ln x\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(3\ln(\sin x)\right)$$

$$\frac{y'}{y} = \frac{1}{4}\cdot\frac{1}{x} + 3\cdot\frac{1}{\sin x}\cdot\cos x$$

Step 3:

$$y' = \left[\frac{1}{4x} + 3\cot x\right]\cdot y$$

$$y' = \left[\frac{1}{4x} + 3\cot x\right]\sqrt[4]{x}\sin^3 x$$

Derivative of Inverse Functions

The trick we have used to compute the derivative of the natural logarithm works in general for inverse functions.

Recall that the **inverse function** $f^{-1}(x)$ of a function $f(x)$ is defined by the property that $f(f^{-1}(x)) = x$.

Warning: Do not confuse $f^{-1}(x)$ with the reciprocal $1/f(x)$.

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

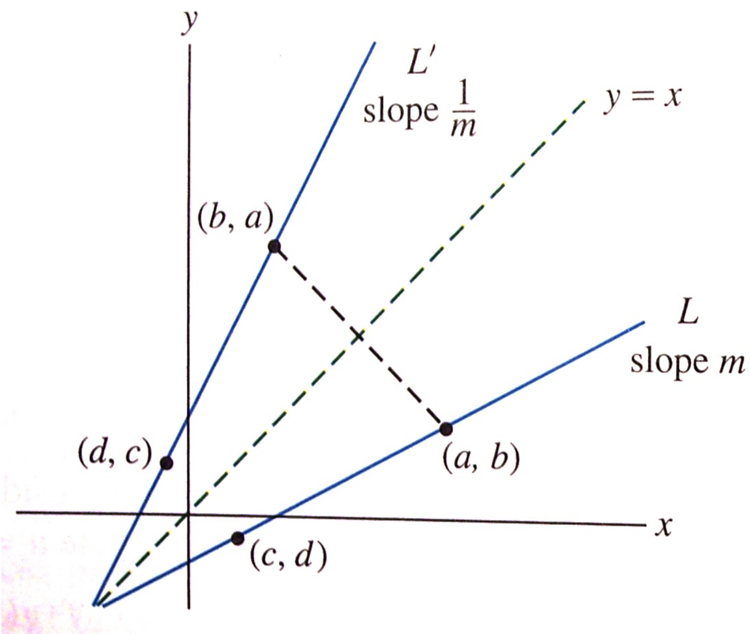
Proof: We differentiate both sides of $f(f^{-1}(x)) = x$:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f^{-1}(x)) = \frac{\mathrm{d}}{\mathrm{d}x} x$$

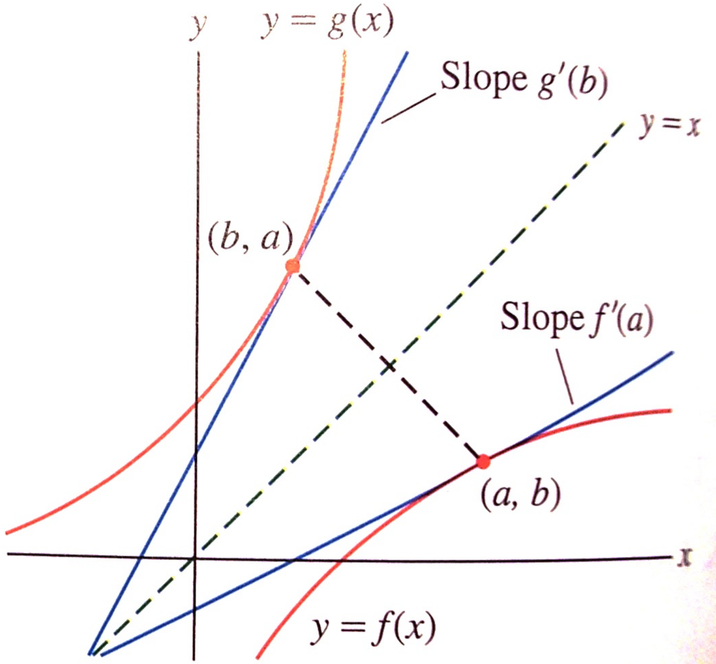
$$f'(f^{-1}(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = 1 \qquad \text{(by the chain rule)}$$

Thus: $$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Graphical Understanding



(A) If L has slope m, then its reflection L' has slope $1/m$.



(B) The tangent line to the inverse $y = g(x)$ is the reflection of the tangent line to $y = f(x)$.

Inverses of the trigonometric functions

$y = \sin^{-1}(x) = \arcsin(x)$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

↖ inverse sine function or arcsine function

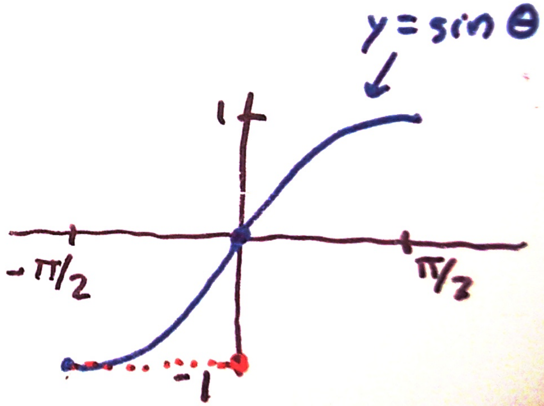
$\sin^{-1}(x)$ = angle (or arc) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

↖ Note: The -1 in the exponent means inverse function, <u>not</u> $\frac{1}{\sin x}$.

<u>ex.</u> $\sin^{-1}(0) = 0$

$\sin^{-1}(-1) = -\pi/2$

$\sin^{-1}(1/\sqrt{2}) = \pi/4$



Inverses of the trigonometric functions

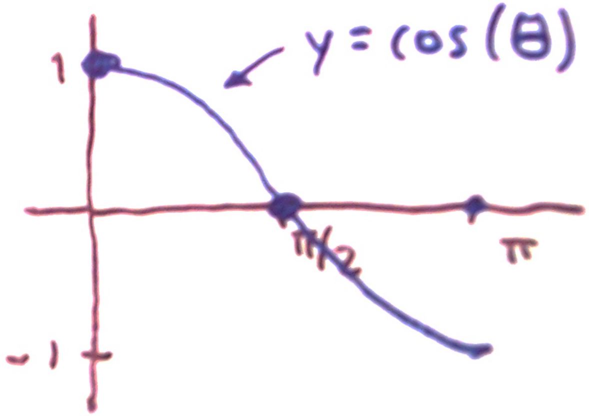
Inverse cosine:

$y = \cos^{-1}(x) = \arccos(x)$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$

$\cos^{-1}(0) = \pi/2$

$\cos^{-1}(1) = 0$

$\cos^{-1}(-1) = \pi$



Example: Find $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x$.

Solution:

$\arcsin x = \sin^{-1} x$ is the inverse function of $\sin x$.

Since $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$, by the theorem for inverse functions:

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\cos(\sin^{-1} x)}$$

Using $\sin^2 z + \cos^2 z = 1$ or $\cos z = \sqrt{1 - \sin^2 z}$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}$$

The arcus tangent function $y = \arctan x$ is defined for $x \in \mathbf{R}$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Example: Find $\dfrac{\mathrm{d}}{\mathrm{d}x} \arctan x = \tan^{-1}(x)$.

Solution:

We use again the theorem for the derivative of inverse functions.

Since $\dfrac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x = \dfrac{1}{\cos^2 x}$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{\sec^2(\tan^{-1} x)} = \cos^2(\tan^{-1} x).$$

Using $\sin^2 z + \cos^2 z = 1$ or $\tan^2 z + 1 = \dfrac{1}{\cos^2 z}$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1 + \tan^2(\tan^{-1} x)} = \frac{1}{1 + x^2}$$

Theorem

$f(x)$	$f'(x)$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\mathrm{arcsec} x$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$