Calculus I - Lecture 10
Trigonometric Functions and the Chain Rule

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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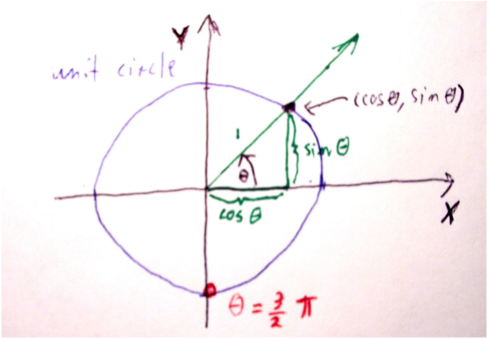
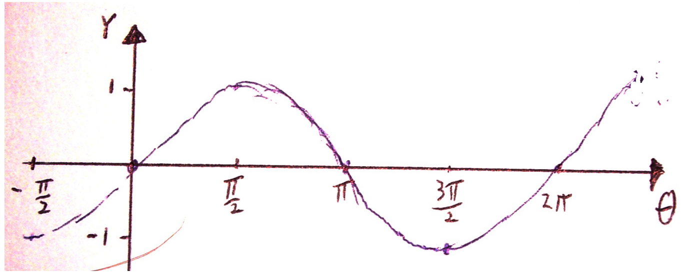
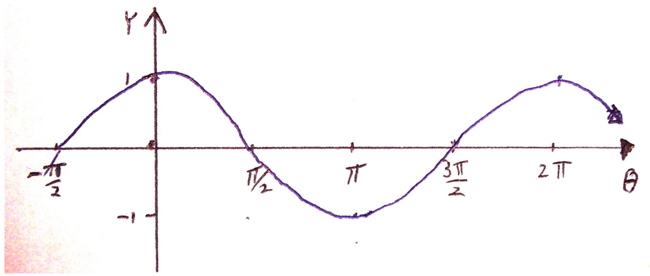
Recall the definition of $\sin\theta$ and $\cos\theta$. 2π radians $= 360°$.



Graph of $\sin\theta$:



Graph of $\cos\theta$:

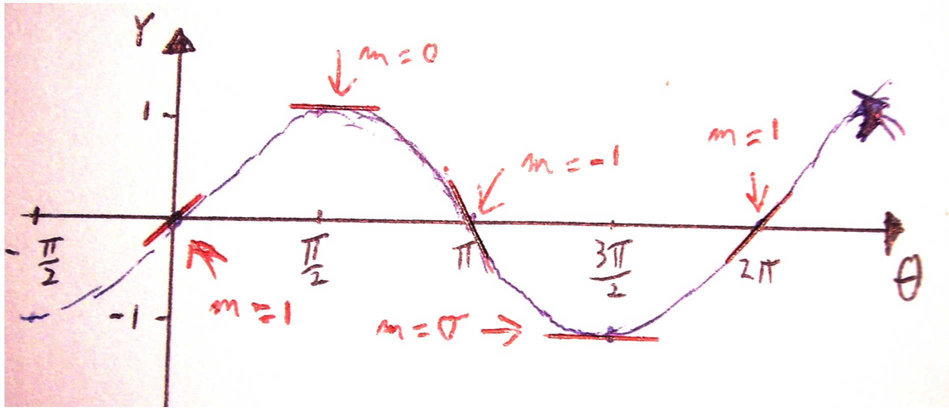


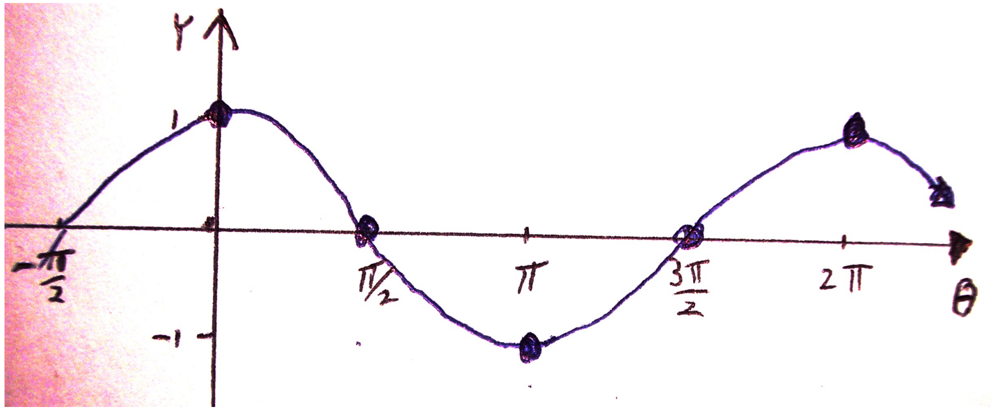
Both are **periodic functions** with period 2π.

Section 3.6 – Trigonometric Derivatives

Find $\frac{\mathrm{d}}{\mathrm{d}x} \sin x$ graphically and algebraically.

Solution: graphically





$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$$

Solution: algebraically

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\ &= \sin x\cdot\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+\cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\ &= \sin x\cdot 0+\cos x\cdot 1 \;=\; \cos x\end{aligned}$$

Where we used

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h-1}{h} &= \lim_{h\to 0}\frac{\sqrt{1-\sin^2 h}-1}{h}\\ &= \lim_{h\to 0}\frac{(\sqrt{1-\sin^2 h}-1)(\sqrt{1-\sin^2 h}+1)}{h\,(\sqrt{1-\sin^2 h}+1)}\\ &= \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{\sqrt{1-\sin^2 h}+1}=1\cdot\frac{0}{\sqrt{1}+1}=0.\end{aligned}$$

Theorem (Derivatives of Trigonometric functions)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}\cos x &= -\sin x \\
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \sec^2 x = \frac{1}{\cos^2 x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\cot x &= -\csc^2 x = -\frac{1}{\sin^2 x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\sec x &= \sec x \tan x \\
\frac{\mathrm{d}}{\mathrm{d}x}\csc x &= -\csc x \tan x
\end{aligned}$$

Here: $\sec^2 x = (\sec x)^2$, etc.

Note the pattern: The derivatives of the "co"-trigonometric functions all have minus (-) signs.

For $\sin x$, we showed already how to get the derivative.
For $\cos x$ this can be done similarly or one uses the fact that the cosine is the shifted sine function.
The remaining trigonometric functions can be obtained from the sine and cosine derivatives.

Example: Find $\frac{\mathrm{d}}{\mathrm{d}x} \tan x$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) \qquad \text{(use quotient rule)}$$

$$= \frac{(\frac{\mathrm{d}}{\mathrm{d}x} \sin x) \cdot \cos x - \sin x \cdot (\frac{\mathrm{d}}{\mathrm{d}x} \cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Section 3.7 – The Chain Rule

Let x, u, y be quantities such that

$$u = g(x) \qquad\qquad y = f(u)$$

change in x $\longrightarrow$ change in u $\longrightarrow$ change in y

$\frac{dy}{dx}$ = change in y with respect to x

$\frac{du}{dx}$ = change in u with respect to x

$\frac{dy}{du}$ = change in y with respect to u

Theorem (Chain Rule)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Leibniz version of the chain rule

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=g(a)} \cdot \left.\frac{du}{dx}\right|_{x=a}$$

The chain rule in function notation

$$u = g(x) \qquad\qquad y = f(u)$$

$$y = f(g(x)) = \text{composition of } f \text{ and } g$$

Chain rule $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$ in Leibniz notation becomes

Theorem (Chain Rule)

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

The derivative of a composition is the derivation of the "outer" function evaluated at the "inner" function **times** the derviative of the "inner" function.

Example: Find $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin(x^2)$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = \cos(x^2) \cdot \frac{\mathrm{d}}{\mathrm{d}x} x^2 = 2x \cos(x^2).$$

Example: Identify the inner and outer functions and find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

a) $y = \sec(5x^2)$.

Solution:

$$y = \sec(5x^2) = f(g(x)),\ f(u) = \sec u \text{ (outer)},\ g(x) = 5x^2 \text{ (inner)}.$$

$$f'(u) = \sec u \cdot \tan u, \quad g'(x) = 10x.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot g'(x)$$

$$= \left[\sec(5x^2)\tan(5x^2)\right] \cdot 10x = 10x\sec(5x^2)\tan(5x^2)$$

b) $y = e^{2x^3}$

Solution:

$$y = e^{2x^3} = f(g(x)), \quad f(u) = e^u \text{ (outer)},\ g(x) = 2x^3 \text{ (inner)}.$$

$$f'(u) = e^u, \qquad g'(x) = 6x^2.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot g'(x)$$

$$= e^{2x^3} \cdot 6x^2 = 6x^2 e^{2x^3}$$

Theorem (General Power Rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(g(x)\big)^n = n\big(g(x)\big)^{n-1}\cdot g'(x)$$

This is a special case of the chain rule. The outer function is $y = u^n$, the inner is $u = g(x)$.

Example: $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[3]{x^3+2x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(x^3+2x\big)^{1/3}$

$$= \tfrac{1}{3}\big(x^3+2x\big)^{\frac{1}{3}-1}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\big(x^3+2x\big)$$

$$= \tfrac{1}{3}\big(x^3+2x\big)^{-2/3}\cdot(3x^2+2)$$

Example: $\frac{\mathrm{d}}{\mathrm{d}x}\sin^5(x-x^2) = \frac{\mathrm{d}}{\mathrm{d}x}\big[\sin(x-x^2)\big]^5$

$$= 5\big[\sin(x-x^2)\big]^4\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sin(x-x^2)$$

$$= 5\big[\sin(x-x^2)\big]^4\cos(x-x^2)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x-x^2)$$

$$= 5\big[\sin(x-x^2)\big]^4\cos(x-x^2)\cdot(1-2x)$$

Example: Compute the derivative of $\sqrt{x^2+\sqrt{x^2+\sqrt{x}}}$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+\sqrt{x^2+\sqrt{x}}}$$

$$=\frac{1}{2\sqrt{x^2+\sqrt{x^2+\sqrt{x}}}}\cdot\left(2x+\frac{1}{2\sqrt{x^2+\sqrt{x}}}\cdot(2x+\frac{1}{2\sqrt{x}})\right)$$

$$=\frac{\frac{2x+\frac{1}{2\sqrt{x}}}{2\sqrt{x^2+\sqrt{x}}}+2x}{2\sqrt{x^2+\sqrt{x^2+\sqrt{x}}}}$$

Homework: Compute the second derivative.

Homework solution

$$\frac{60x^{7/2} + 42x^2 - 7\sqrt{x^2 + \sqrt{x}} + 4\sqrt{x^2 + \sqrt{x}}x^{3/2} - 16\sqrt{x^2 + \sqrt{x}}x^3}{64x^{3/2}\left(x^{3/2} + 1\right)\sqrt{x^2 + \sqrt{x}}\left(x^2 + \sqrt{x^2 + \sqrt{x}}\right)^{3/2}}$$